

On causal band-limited mean square approximation

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Abstract

We study causal dynamic approximation of non-bandlimited processes by band-limited processes such that a part of the historical path of the underlying process is approximated in L_2 -norm by the trace of a band-limited process. This allows to cover the case of irregular non-smooth processes. We show that this problem has an unique optimal solution. The approximating band-limited process has unique extrapolation on future times and can be interpreted as a optimal forecast. To accommodate the current flow of observations, the selection of this band-limited process has to be changed dynamically. This can be interpreted as a causal and linear filter that is not time invariant.

Key words: band-limited processes, causal filters, sampling, low-pass filters, prediction.

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1 Introduction

We study causal dynamic approximation of non-bandlimited processes by band-limited processes. It is known that it is not possible to find an ideal low-pass causal linear time-invariant filter. It is also known that the distance of the set of these ideal low-pass time invariant filters from the set of all causal filters is positive [1]. In addition, it is known that optimal approximation of the ideal low-pass filter is not feasible in the class of causal linear time-invariant

filters (see, e.g., [3] and references here). In the present paper, we are trying to substitute the solution of these unsolvable problems by solution of an easier problem where the filter is not necessary time invariant. Our motivation is that, for some problems, time invariancy for a filter is not crucial. For example, a typical approach to forecasting in finance is to approximate the known path of the stock price process by a smooth process that has an unique extrapolation and accept this extrapolation as the forecast. This procedure has to be done at current time; it is not required that the same forecasting rule will be applied at future times. We apply this approach with the band-limited processes used as approximating smooth predictable processes. More precisely, we suggest to approximate in L_2 -norm the known historical path of the process by the trace of a band-limited process. In this setting, the approximating curve does not necessarily match the underlying process at given sampling points. This is different from classical sampling approach (see, e.g., [7]). Similarly to [4]-[5], our setting allows to cover the case of irregular non-differentiable or discontinuous processes such as historical stock prices in continuous time models. The difference is that [4]-[5] achieves point-wise matching for the underlying process being smoothed by a convolution operator; we consider approximation of the underlying process directly using different methods. In [4]-[5], the estimate of the error norm is given. In our setting, it is guaranteed that the approximation generates the error of the minimal norm.

We show that an unique optimal solution of approximation problem exists. The approximating process is derived in time domain in a form of sinc series. To accommodate the current flow of observations, the coefficients of these series and the related band-limited processes have to be changed dynamically. It can be interpreted as a causal and linear filter that is not time invariant.

2 Definitions

We denote by $L_2(D)$ the usual Hilbert space of complex valued square integrable functions $x : D \rightarrow \mathbf{C}$, where D is a domain.

For $x(\cdot) \in L_2(\mathbf{R})$, we denote by $X = \mathcal{F}x$ the function defined on $i\mathbf{R}$ as the Fourier transform of $x(\cdot)$;

$$X(i\omega) = (\mathcal{F}x)(i\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} x(t) dt, \quad \omega \in \mathbf{R}.$$

Here $i = \sqrt{-1}$. For $x(\cdot) \in L_2(\mathbf{R})$, the Fourier transform X is defined as an element of $L_2(\mathbf{R})$

(more precisely, $X(i\cdot) \in L_2(\mathbf{R})$).

For a given $\Omega > 0$, let $\mathcal{U}_{\Omega,\infty} = \{X(i\omega) \in L_2(i\mathbf{R}) : X(i\omega) = 0 \text{ for } |\omega| > \Omega\}$, and let $\mathcal{U}_{\Omega,N}$ be the set of all $X \in \mathcal{U}_{\Omega,\infty}$ such that there exists a sequence $\{y_k\}_{k=-N}^N \in \mathbf{C}^{2N+1}$ such that $X(i\omega) = \sum_{k=-N}^N y_k e^{ik\omega/\Omega} \mathbb{I}_{\{|\omega| \leq \Omega\}}$, where \mathbb{I} is the indicator function.

For $N = +\infty$ and for integers $N \geq 0$, consider Hilbert spaces \mathcal{Y}_N such that $\mathcal{Y}_N = \mathbf{C}^{2N+1}$ for $N < +\infty$ and \mathcal{Y}_N is the set of all sequences $\{y_k\}_{k=-N}^N \in \mathbf{C}^{2N+1}$ such that $\sum_{k=-\infty}^{\infty} |c_k|^2 < +\infty$.

Let $s \in \mathbf{R}$ and $q < s$ be given; the case when $q = -\infty$ is not excluded. Consider Hilbert spaces of complex valued functions $\mathcal{X} = L_2(-\infty, +\infty)$ and $\mathcal{X}_- = L_2(q, s)$.

Let $\Omega > 0$ and N be given (the case of $N = +\infty$ is not excluded). Let $\mathcal{X}_{\Omega,N}$ be the subset of \mathcal{X}_- consisting of functions $x|_{(q,s]}$, where $x \in \mathcal{X}$ are such that $x(t) = (\mathcal{F}^{-1}X)(t)$ for $t \in [q, s]$ for some $X(i\omega) \in \mathcal{U}_{\Omega,N}$.

Proposition 2.1 *For any $x \in \mathcal{X}_{\Omega,N}$, there exists a unique $X \in \mathcal{U}_{\Omega,N}$ such that $x(t) = (\mathcal{F}^{-1}X)(t)$ for $t \in [q, s]$.*

For a Hilbert space H , we denote by $(\cdot, \cdot)_H$ the corresponding inner product. We use notation $\text{sinc}(x) = \sin(x)/x$.

3 Main results

3.1 Optimal band-limited approximation

Let $x \in \mathcal{X}$ be a process. We assume that the path $x(s)|_{s \in [q,s]}$ represents available historical data. Let Hermitian form $F : \mathcal{X}_{\Omega,N} \times \mathcal{X}_- \rightarrow \mathbf{R}$ be defined as

$$F(\hat{x}, x) = \int_q^s |\hat{x}(t) - x(t)|^2 dt.$$

Theorem 3.1 *For any $N \leq +\infty$, there exists a unique solution \hat{x} of the minimization problem*

$$\text{Minimize} \quad F(\hat{x}, x) \quad \text{over} \quad \hat{x} \in \mathcal{X}_{\Omega,N}. \quad (3.1)$$

Remark 3.1 *By Proposition 2.1, there exists a unique extrapolation of the band-limited solution $\hat{x}(t)$ of problem (3.1) on the future time interval $(s, +\infty)$. It can be interpreted as the optimal forecast (optimal given Ω and N).*

3.2 Optimal sinc coefficients

To solve problem (3.1) numerically, it is convenient to expand $X(i\omega)$ via Fourier series.

For a given $\Omega > 0$, consider the mapping $\mathcal{Q} : \mathcal{Y}_N \rightarrow \mathcal{X}_{\Omega, N}$ such that $x = \mathcal{Q}y$ is such that $x(t) = (\mathcal{F}^{-1}X)(t)$ for a.e. $t \in (q, s]$, where

$$X(i\omega) = \sum_{t=-N}^N y_t e^{it\omega/\Omega} \mathbb{I}_{\{|\omega| \leq \Omega\}}.$$

Clearly, this mapping is linear and continuous.

Let Hermitian form $G : \mathcal{Y}_N \times \mathcal{X}_- \rightarrow \mathbf{R}$ be defined as

$$G(y, x) = F(\mathcal{Q}y, x) = \int_q^s |\hat{x}(t) - x(t)|^2 dt, \quad \hat{x} = \mathcal{Q}y. \quad (3.2)$$

Corollary 3.1 *For any $N \leq +\infty$, there exists a unique solution y of the minimization problem*

$$\text{Minimize} \quad G(y, x) \quad \text{over} \quad y \in \mathcal{Y}_N. \quad (3.3)$$

Problem (3.1) can be solved via problem (3.3); its solution with $N < +\infty$ can be found numerically.

3.3 Solution of problem (3.3)

Let N be given, let Z be the set of all integers z such that $|z| \leq N$ if $N < +\infty$, and let Z be the set of all integers if $N = +\infty$. Let

$$X(i\omega) = \sum_{k \in Z} y_k e^{ik\omega\pi/\Omega} \mathbb{I}_{\{|\omega| \leq \Omega\}},$$

where $\{y_k\} \in \mathcal{Y}_N$. Let $\hat{x} = \mathcal{F}^{-1}X$. We have that

$$\begin{aligned} \hat{x}(t) &= \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \left(\sum_{k \in Z} y_k e^{ik\omega\pi/\Omega} \right) e^{i\omega t} d\omega = \frac{1}{2\pi} \sum_{k \in Z} y_k \int_{-\Omega}^{\Omega} e^{ik\omega\pi/\Omega + i\omega t} d\omega \\ &= \frac{1}{2\pi} \sum_{k \in Z} y_k \frac{e^{ik\pi + i\Omega t} - e^{-ik\pi - i\Omega t}}{ik\pi/\Omega + it} = \frac{\Omega}{\pi} \sum_{k \in Z} y_k \text{sinc}(k\pi + \Omega t). \end{aligned}$$

Remark 3.2 Let $t[k] = -k\pi/\Omega$. Clearly, $\hat{x} = \mathcal{F}^{-1}X$ is such that $\hat{x}(t[k]) = y_k \cdot \Omega/\pi$, i.e., $y_k = \hat{x}(t[k]) \cdot \pi/\Omega$, and, therefore,

$$\hat{x}(t) = \sum_{k \in \mathbb{Z}} \hat{x}(t[k]) \text{sinc}(k\pi + \Omega t).$$

It gives celebrated Sampling Theorem; see, e.g., [7].

Remark 3.3 We consider a setting when only the part $x(t)|_{t \in [q, s]}$ of the path of the process is available at current time $s < +\infty$. In this setting, sampling theorem is not applicable. Our approximation can be considered as a modification of the truncated sinc approximation (see, e.g., [6], [7]). The difference is that the increasing of N is not related to extension the time interval $[q, s]$ in our setting.

We have that

$$\begin{aligned} G(y, x) &= \int_q^s |\hat{x}(t) - x(t)|^2 dt = \int_q^s \left| \frac{\Omega}{\pi} \sum_{k \in \mathbb{Z}} y_k \text{sinc}(k\pi + \Omega t) - x(t) \right|^2 dt \\ &= (y, Ry)_{\mathcal{Y}_N} - 2\text{Re}(y, rx)_{\mathcal{X}_-} + (\rho x, x)_{\mathcal{X}_-}. \end{aligned} \quad (3.4)$$

Here $R : \mathcal{Y}_N \times \mathcal{Y}_N \rightarrow \mathcal{Y}_N$ is a linear bounded Hermitian operator, $r : \mathcal{X}_- \rightarrow \mathcal{Y}_N$ is a bounded linear operator, $\rho : \mathcal{X}_- \times \mathcal{X}_- \rightarrow \mathcal{X}_-$ is a linear bounded Hermitian operator.

It follows from the definitions that the operator R is non-negatively defined (it suffices to substitute $x(t) \equiv 0$ into the Hermitian form).

3.4 The case when $N < +\infty$

Up to the end of this paper, we assume that $N < +\infty$. In this case, the space \mathcal{Y}_N is finite dimensional, it follows that the operator R can be represented via a matrix $R = \{R_{km}\} \in \mathbb{C}^{2N+1, 2N+1}$, where $R_{km} = \bar{R}_{mk}$ and $(Ry)_k = \sum_{m=-N}^N R_{km} y_m$.

Theorem 3.2 (i) For any $N < +\infty$, the operator R is positively defined.

(ii) Problem (3.3) has a unique solution $\hat{y} = R^{-1}rx$.

(iii) The components of the matrix R can be found from the equality

$$R_{km} = \frac{\Omega^2}{\pi^2} \int_q^s \text{sinc}(m\pi + \Omega t) \text{sinc}(k\pi + \Omega t) dt. \quad (3.5)$$

(iv) The components of the vector $rx = \{(rx)_k\}_{k=-N}^N$ can be found from the equality

$$(rx)_k = \frac{\Omega}{\pi} \int_q^s \text{sinc}(k\pi + \Omega t) x(t) dt. \quad (3.6)$$

Corollary 3.2 Let $\hat{y} = \hat{y}(q, s)$ be the vector calculated as in Theorem 3.2, $\hat{y} = \{\hat{y}_k\}_{k=-N}^N$. The process

$$\hat{x}(t) = \hat{x}(t, q, s) = \frac{\Omega}{\pi} \sum_{k \in Z} y_k \text{sinc}(k\pi + \Omega t)$$

represents the output of a causal filter that is linear but not time invariant.

4 Numerical experiments

In the numerical experiments described below, we have used MATLAB symbolic integration for calculation of integrals (3.5) and (3.6). The experiments show that some eigenvalues of R are quite close to zero. Because of the integration errors, some eigenvalues of the calculated matrix R are actually fluctuating around zero despite the fact that, by Theorem 3.2, $R > 0$. Respectively, the error $E = \|R\hat{y} - rx\|_{L_2(q,s)}$ for the MATLAB solution of the equation $R\hat{y} = rx$ does not vanish. This error depends on the error tolerance parameter tol of MATLAB integration operator $QUAD$ that was used; the default value is $tol = 10^{-6}$; we used $tol = 10^{-8}$. Further, in our experiments, we found that the error E can be decreased by the replacing R in the equation $\hat{x} = R^{-1}rx$ by $R_\varepsilon = R + \varepsilon I$, where I is the unit matrix and where $\varepsilon > 0$ is small. In particular, for $\varepsilon = 0.001$, the corresponding error $E(\varepsilon) = \|R_\varepsilon^{-1}rx - \hat{y}\|_{L_2(q,s)} < \|R^{-1}rx - \hat{y}\|_{L_2(q,s)}$, i.e., the approximation on $[q, s]$ is better for $\hat{y} = R_\varepsilon^{-1}rx$ calculated for $\varepsilon = 0.001$ than for $\hat{y} = R^{-1}rx$ calculated for $\varepsilon = 0$.

Figures 5.1 and 5.2 show examples of a process $x(t)$ and the band-limited process $\hat{x}(t)$ approximating $x(t)$ on time intervals $(q, s] = (-12, -2]$ and $(q, s] = (-10, 0]$, respectively, calculated with $\varepsilon = 0.001$ for $\Omega = 4$ and $N = 30$. As expected, the change of the time interval from $(q, s] = (-12, -2]$ to $(q, s] = (-10, 0]$ results in the change of the approximating band-limited process.

Note that the experiments demonstrate robustness with respect to the changes of N . The curves of $\hat{x}(t)$ will be almost the same if we consider $N = 50$ instead of $N = 30$, when all other

parameters are the same. However, the error E is larger for large $N = 100$, due to accumulated larger error of integration.

The shape of curves of $\hat{x}(t)$ depends on the choice Ω . Figure 5.3 shows an example of a process $x(t)$ and of the band-limited process $\hat{x}(t)$ approximating $x(t)$ on time interval $(q, s] = (-10, 0]$ calculated for $\Omega = 2$, when all other parameters are the same as for Figure 5.2.

By Remark 3.1, the extrapolation of the process $\hat{x} \in \mathcal{X}_{\Omega, N}$ on the future time interval $(s, +\infty)$ can be interpreted as the optimal forecast (optimal given Ω and N).

Remark 4.1 We have used the procedure of replacement R by $R_\varepsilon = R + \varepsilon I$ with small $\varepsilon > 0$ to reduce the error of calculation of the inverse matrix for the matrix R that is positively defined but is close to a degenerate matrix. It can be noted that the same replacement could lead to a meaningful setting for the case when $\varepsilon > 0$ is not small. More precisely, it leads to optimization problem

$$\text{Minimize} \quad G(y, x) + \varepsilon^2 \sum_{k=-N}^N |y_k|^2 \quad \text{over} \quad y \in \mathcal{Y}_N. \quad (4.1)$$

The solution restrains the norm of y , and, respectively, the norm of \hat{x} .

Figure 5.4 illustrates Remark 4.1 with an example of a process $x(t)$ and the corresponding band-limited process $\hat{x}(t)$ calculated via solution of problem (4.1) for $\varepsilon = 0.05$, when all other parameters are the same as for Figure 5.2. This solution was obtained by replacement of R by $R_\varepsilon = R + \varepsilon I$ with $\varepsilon = 0.05$.

5 Appendix: proofs

Proof of Proposition 2.1. The statement of this proposition is known in principle. It suffices to prove that if $x(\cdot) \in \mathcal{X}_{\Omega, N}$ is such that $x(t) = 0$ for $t \in (q, s]$, then $x(t) \equiv 0$. For the sake of completeness, we give below a proof. For $C > 0$, consider a class $\mathcal{M}(C)$ of infinitely differentiable functions $x(t) : \mathbf{R} \rightarrow \mathbf{R}$ such that there exists $M = M(x(\cdot)) > 0$ such that

$$\left\| \frac{d^k x}{dt^k}(\cdot) \right\|_{L_2(\mathbf{R})}^2 \leq C^k M, \quad k = 0, 1, 2, \dots$$

Let $\mathcal{M} = \cup_{C>0} \mathcal{M}(C)$. Any $x \in \mathcal{M}$ is infinitely differentiable and such that there exists $C_1 = C_1(x(\cdot)) > 0$ and $M_1 = M_1(x(\cdot)) > 0$ such that

$$\sup_t \left| \frac{d^k x}{dt^k}(t) \right| \leq C_1^k M_1.$$

Clearly, $\mathcal{X}_{\Omega,N} \subset \mathcal{M}$. Therefore, any $x(\cdot) \in \mathcal{X}_{\Omega,N}$ is analytic and allows the Taylor series expansion at any point with an arbitrarily large radius of convergence. Consider the Taylor series expansion at $t_0 \in (q, s)$. Since all derivatives at this point are equal to zero, the expansion is identically equal to zero. This completes the proof of Proposition 2.1. \square

Proof of Theorem 3.1. It suffices to prove that $\mathcal{X}_{\Omega,N}$ is a closed linear subspace of $L_2(q, s)$. In this case, there exists a unique projection \hat{x} of $x|_{[q,s]}$ on $\mathcal{X}_{\Omega,N}$, and the theorem is proven.

Clearly, for any $N \leq +\infty$, the set $U_{\Omega,N}$ is a closed linear subspace of $L_2(\mathbf{R})$. Consider a mapping $Q : \mathcal{U}_{\Omega,N} \rightarrow \mathcal{X}_{\Omega,N}$ such that $x(t) = (QX)(t) = (\mathcal{F}^{-1}X)(t)$ for $t \in [q, s]$. It is a linear continuous operator. By Proposition 2.1, it is a bijection. Since this mapping is continuous, it follows that the inverse mapping $Q^{-1} : \mathcal{X}_{\Omega,N} \rightarrow U_{\Omega,N}$ is also continuous (see Corollary in Ch.II.5 [8], p.77). Since the set $U_{\Omega,N}$ is a closed linear subspace of $L_2(\mathbf{R})$, it follows that $\mathcal{X}_{\Omega,N}$ is a closed linear subspace of \mathcal{X}_- . This completes the proof of Theorem 3.1. \square

Proof of Theorem 3.2. Let us prove statement (i). We know that $R \geq 0$. Suppose that there exists $\bar{y} \in \mathbf{C}^{2N+1}$ such that $\bar{y} \neq 0$ and $R\bar{y} = 0$. Let $r^* : \mathcal{Y}_N \rightarrow \mathcal{X}_-$ be the adjoint operator to the operator $r^* : \mathcal{X}_- \rightarrow \mathcal{Y}_N$. If $r^*\bar{y} \neq 0$ then there exists $x \in \mathcal{X}_-$ such that $G(\bar{y}, x) < 0$, which is not possible since $G(y, x) \geq 0$ for all y, x . Therefore, $r^*\bar{y} = 0$, i.e., $G(\bar{y}, x) = (\rho x, x)_{\mathcal{X}_-}$. Further, let \hat{y} be a solution of problem (3.3). We have that $G(\hat{y}, x) = G(\hat{y} + \bar{y}, x)$. Hence $\hat{y} + \bar{y} \neq \hat{y}$ is another solution of problem (3.3). This contradicts to Corollary 3.1 that states that this problem has an unique solution. Statement (ii) follows from (i) and from classical theory of quadratic forms. Statements (iii)-(iv) follow immediately from representation (3.4). This completes the proof of Theorem 3.2. \square

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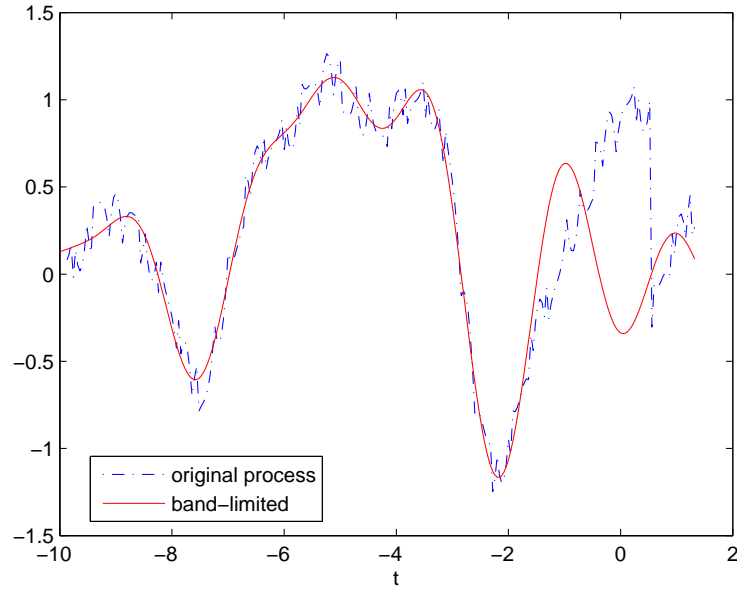


Figure 5.1: Example of $x(t)$ and band-limited process $\hat{x}(t)$ approximating $x(t)$ on $(q, s] = (-12, -2]$, with $\Omega = 4$, and $N = 30$.

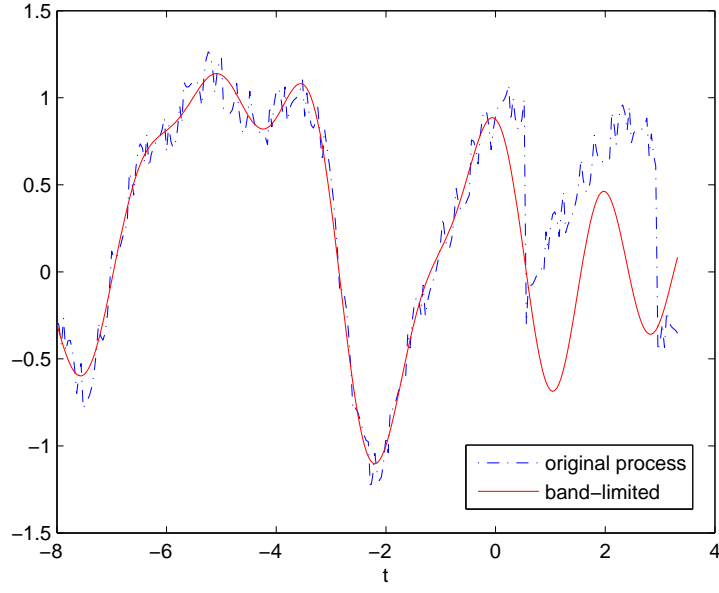


Figure 5.2: Example of $x(t)$ and band-limited process $\hat{x}(t)$ approximating $x(t)$ on $(q, s] = (-10, 0]$, with $\Omega = 4$, and $N = 30$.

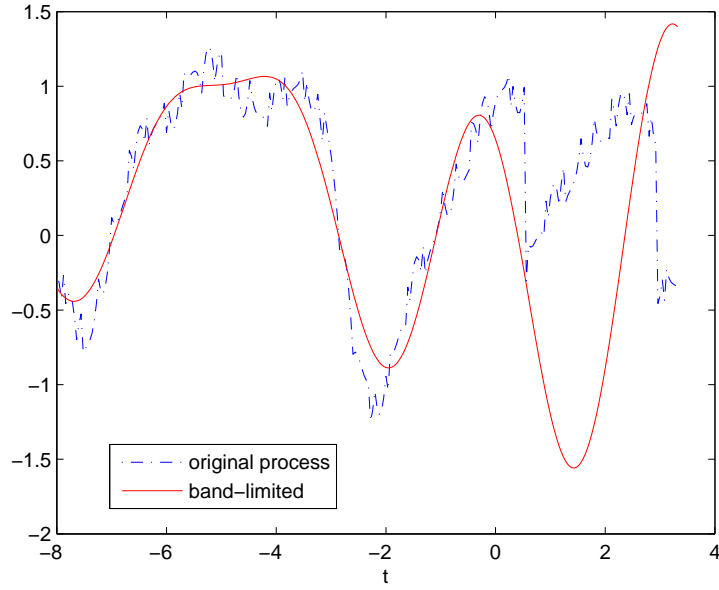


Figure 5.3: Example of $x(t)$ and band-limited process $\hat{x}(t)$ approximating $x(t)$ on $(q, s] = (-10, 0]$, with $\Omega = 2$, and $N = 30$.

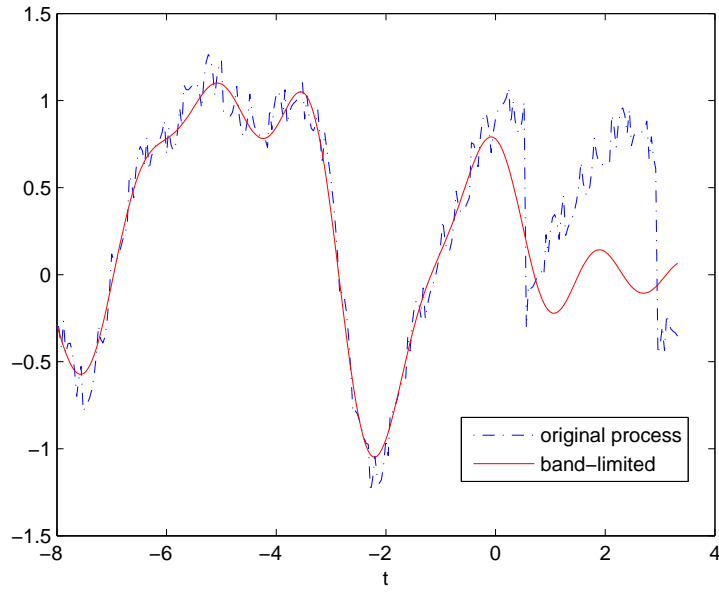


Figure 5.4: Example of $x(t)$ and band-limited process $\hat{x}(t)$ calculated via solution of problem (4.1) for $\varepsilon = 0.05$, $(q, s] = (-10, 0]$, $\Omega = 4$, and $N = 30$.